JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **115**, 506–516 (1986)

Canonical Equivalence of Integrable Systems, Their Associated (Semi)-Groups, and Moment Systems

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By exhibiting the canonical transformation that does the job, we prove that all integrable systems on $\mathbb{R}^{2n}$ are canonically equivalent. This equivalence has a unitary counterpart in the quantum mechanical set up, and a nonunitary counterpart transforming the semigroups associated to classical systems. It can also be used to transform the monet system moment and cross-sequences associated to each Hamiltonian onto each other. © 1986 Academic Press, Inc

1. INTRODUCTION

In this paper we do several things. One quite simple, but seemingly unnoticed so far: a proof of the canonical equicalence of integrable Hamiltonian systems. The other is using a theory of representations of these transformations we prove that the semigroups associated to the Hamiltonian systems can be transformed onto each other. Associated to each Hamiltonian there are several polynomial families that can be transformed onto each other as well.

This last problem was approached from a "static" point of view in [11], where a time-independent class of canonical transformations was used to deal with another approach to umbral calculus.

In Section 2 we do the basics of Hamiltonian mechanics on $\mathbb{R}^{2n}$, we construct the canonical transformation relating only two integrable systems, and we shall see that when the Hamilton–Jacobi function and its inverse exist, the systems must be integrable.

A set of references for section 2 could be [1, 8, 11, 19]. In Section 3 the equivalence of the canonical setups is implemented unitarily, and thus the unitary equivalence of quantum mechanics is established. As sources with the basics use [10] or [13].

In Sections 4 and 5 we follow [3] or its fortcoming expanded version [4] for the connection between probability, hamiltonian systems and

0022-247X/86 $3.00

evolution semigroups. This setup was extended a bit in [6] where the possibility of using nonunitary representations to transform evolution semigroups was explored.

2. Canonical Equivalence of Integrable Systems on $\mathbb{R}^{2n}$

A Hamiltonian system on $\mathbb{R}^{2n}$ is specified by giving a function $H(q, p)$: $\mathbb{R}^{2n} \to \mathbb{R}$. The trajectories of the system are to be obtained solving the (Hamiltonian) set

$$\begin{aligned} \dot{q} &= \partial H/\partial p_i, \\ \dot{p}_i &= -\partial H/\partial p_i \qquad (q(0), p(0)) = (q_0, p_0). \end{aligned} \tag{2.1}$$

A system is said to be integrable when its Hamiltonian is a function of the momenta, i.e., the p variables only. Consider two such systems. To distinguish between them, denote their coordinates by $(q, p) \equiv (q_1, ..., q_n, p_1, ..., p_n)$ and $(Q, P) \equiv (Q_1, ..., Q_n, P_1, ..., P_n)$, respectively, and let $H(p)$ and $\tilde{H}(P)$ be their Hamiltonians.

The equations of motion (2.1) can be now trivially solved yielding

$$\begin{aligned} q(t) &= q_0 + v(p_0)\, t, \qquad & Q(t) &= Q_0 + \tilde{v}(P_0)\, t, \\ p(t) &= p_0 - , & P(t) &= P_0, \end{aligned} \tag{2.2}$$

where $v(p) = \nabla H(p)$, $\tilde{v}(P) = \nabla \tilde{H}(P)$, ∇ denoting the gradient with respect to the obvious variables.

Let us now verify that the functions

$$F(q, P, t) = q \cdot P - tH(P) + t\tilde{H}(P), \tag{2.3a}$$

$$F'(Q, p, t) = Q \cdot p + tH(p) - t\tilde{H}(p) \tag{2.3b}$$

are generating functions of the second type (F_2 in the standard notation) transforming one system onto the other.

Caution. *The coordinate q in* (2.3a) *is to be taken at the coordinate at present time t for the "initial" system and* the momentum P is to be *taken as momentum at time t for the* "final" system. (To be really proper, extended phase-space should be used. See [2] or [14].)

The dot in (2.3) indicates scalar product in $\mathbb{R}^n$.

The transformation equations for (2.3a) are

$$Q = \nabla_p F = q - t\nabla H(P) + t\nabla \tilde{H}(P), \tag{2.4a}$$

$$p = \nabla_q F = P, \tag{2.4b}$$

$$\tilde{H} = H + \partial F/\partial t, \tag{2.4c}$$

and the corresponding set for (2.3b) is

$$q = \nabla_p F' = Q + t\nabla H(p) - t\nabla \tilde{H}(p), \tag{2.5a}$$

$$P = \nabla_Q F = p, \tag{2.5b}$$

$$H = \tilde{H} + \partial F'/\partial t. \tag{2.5c}$$

We shall now give a composition rule for generating functions which shall make the meaning of (2.3) obvious in terms of (2.2), and which we shall use below for the extension of these results. The next lemma was introduced in [6], but its proof is very easy.

LEMMA 2.6. *Let $F^1(q_1, p_2, t)$ and $F^2(q_2, p_3, t)$ be the generating functions, of the second type, of the canonical transformations $(q_1, p_1) \to (q_2, p_2 \to (q_3, p_3)$, respectively. Then*

$$F(q_1, p_3, t) = F^1(q_1, p_2, t) - q_2 \cdot p_2 + F^2(q_2, p_3, t)$$

generates the composite transformation $(q_1, p_1) \to (q_3, p_3)$.

COMMENT. q_2 and p_2 have to be eliminated above using

$$q_2 = \nabla_{p_2} F^1 \qquad \text{and} \qquad p_2 = \nabla_{q_2} F^2.$$

Proof.

$$\nabla_{q_1} F = \nabla_{q_1} F^1 = p_1, \qquad \nabla_{p_3} F = \nabla_{p_3} F^2 = q_3$$

and

$$\partial F/\partial t = \partial F^1/\partial t + \partial F^2/\partial t = H_2 - H_1 + H_3 - H_2 = H_3 - H_1$$

or

$$H_3 = H + \partial F/\partial t.$$

Now consider, a Hamiltonian system in $\mathbb{R}^{2n}$ with Hamiltonian $H(q, p)$ and assume that the following two initial value problems have solutions for all t.

$$H\left(\bar{q}, \frac{\nabla S}{q}\right) + \frac{\partial S}{\partial t} = 0, \tag{2.7a}$$

$$S(\bar{q}, p, 0) = \bar{q} \cdot p, \tag{2.7b}$$

$$H(q, \nabla_q \bar{S}) = \frac{\partial \bar{S}}{\partial t}, \tag{2.8a}$$

$$\bar{S}(q, \bar{p}, 0) = q \cdot \bar{p}. \tag{2.8b}$$

Then the functions $S(\bar{q}, p, t)$ and $\bar{S}(q, \bar{p}, t)$ are inverse to each other according to the composition law of Lemma 2.6, and generate, respectively, the canonical transformation taking the system from time t to rest and the transformation taking the system from rest to time t.

In each case the hamiltonian at rest is 0 and time t is $H(q, p)$. The essential thing in (2.7) and (2.8) is that at $t = 0$ both S and $\bar{S}$ generate the identity transformation. S is called the Hamilton–Jacobi function. We now state

THEOREM 2.9. *Assume that S and $\bar{S}$ satisfying* (2.7) *and* (2.8) *exist, and let $\tilde{H}(P)$ be the Hamiltonian of any other integrable system. Then*

$$F(\bar{q}), P, t) = S(\bar{q}, p, t) - q \cdot p + \tilde{F}'(q, P, t), \tag{2.10a}$$

$$F'(Q, \bar{p}, t) = \tilde{F}(Q, p, t) - q \cdot p + \bar{S}(q, \bar{p}, t), \tag{2.10b}$$

where $\tilde{F}(Q, p, t) = Q \cdot p - tH(p)$ and $\tilde{F}'(q, P, t) = q \cdot P + t\tilde{H}(P)$.

COMMENT. $\tilde{F}$ and $\tilde{F}'$ are the Hamiltonian–Jacobi function and its inverse for the system with Hamiltonian $\tilde{H}(P)$ Thus (2.3) become a particular case of (2.10).

Proof. Apply Lemma 2.6.

Thus we see that integrability is the same thing as global existence of solutions to (2.7) and (2.8). As an example, consider the Hamilton–Jacobi function and its inverse, given by

$$S(\bar{q}, p, t) = \frac{\bar{q} \cdot p}{\cos t} - \frac{\bar{q}^2 + p^2}{2} \tan t, \tag{2.11a}$$

$$\bar{S}(q, \bar{p}, t) = \frac{q \cdot \bar{p}}{\cos t} + \frac{q^2 + \bar{p}^2}{2} \tan t, \tag{2.11b}$$

which are easily verified to satisfy $\frac{1}{2}(\nabla S)^2 + \frac{1}{2}q^2 + (\partial S/\partial t) = 0$ and $\frac{1}{2}(\nabla_q \bar{S})^2 + \frac{1}{2}q^2 = \partial \bar{S}/\partial t$. Thus according to (2.9),

$$\begin{aligned} F(\bar{q}, P, t) &= S(\bar{q}, p, t) - q \cdot p + \left(q \cdot P + \frac{tP^2}{2}\right) \\ &= S(\bar{q}, P, t) + \frac{tP^2}{2} \end{aligned}$$

after using $p=\nabla_q(q\cdot P+(tP^2/2))$, generates a canonical transformation mapping the harmonic oscillator onto a free particle. Again, since we are using a space-time picture, trajectories go onto trajectories and there is no problem with the fact that the orbits of the oscillator in phase-space are closed and those of the free particle are open.

Some examples, relevant to the probabilistic set-up (see Sect. 4) are given by

$$\begin{aligned} H_1 &= -a\cdot p, \\ H_2 &= \tfrac{1}{2}p^2, \\ H &= \int (e^{-\xi\cdot p}-1)\,\mu(d\xi), \end{aligned} \tag{2.12}$$

where a is a constant vector and $\mu(d\xi)$ is a finite measure on $\mathbb{R}^n$.

The transformation functions between these cases are just a transcription of (2.3) by specialization of the symbols.

3. Unitary Representations: Quantum Mechanical Case

Canonical transformations of the type (2.3) are easily implemented by unitary transformations mapping the Hilbert spaces associated to each system onto each other.

The procedure to find the transformations is to rewrite the transformtion equations (2.4) with the classical variables replaced by their corresponding operators.

$$\hat{Q}=\hat{q}-t\nabla H(p)+t\hat{H}(p), \tag{3.1a}$$

$$\hat{P}=\hat{p}, \tag{3.1b}$$

where $\hat{q}$ denotes the operator of multiplication by q and $\hat{p}$ the operator $-i\nabla_q$. It is easy to convince onself that

$$\langle q\,|\,P)_t=(2\pi)^{n/2}\exp iF(q,P,t) \tag{3.2}$$

is the matrix element of the unitary transformation to go from the $|q\rangle$ basis to the $|P)$ basis. For this it suffices to verify that (3.2) satisfies the transformation equations (see [6, 7]).

$$\begin{aligned} \langle q\,|\,\hat{p}\,|\,P) &= -i\frac{\partial}{\partial q}\langle q\,|\,P)=P\langle q\,|\,P), \\ \langle q\,|\,\hat{Q}\,|\,P) &= i\frac{\partial}{\partial p}\langle q\,|\,P)=(q-tv(P)+t\tilde{v}(P))\langle q\,|\,P). \end{aligned} \tag{3.3}$$

It is easy to see that $\langle q | q' \rangle = \int \langle q | P)(P | q' \rangle \, dP = \delta(q - q')$ and $(P | P') = \int (P | q \rangle \langle q | P') \, dq = \delta(P - P')$. It is also simple to verify that if $\tilde{\psi}(Q, t)$ satisfies

$$i \frac{\partial \tilde{\psi}}{\partial t} = \tilde{H}(-i\nabla_Q) \, \tilde{\psi}$$

plus initial conditions, then $\psi(q, t) = \int \langle q | P) \, \tilde{\psi}(P, t) \, dP$, with

$$\tilde{\psi} = \int e^{-iQ \cdot P} \psi(Q, t) \, dQ/(2\pi)^{n.2}$$

satisfies

$$i \frac{\partial \psi}{\partial t} = H(-i\nabla_q) \, \psi$$

plus the same initial conditions as ψ since at $t = 0$ all transformations coincide with the identity. Using the results in [10] or [13] and (2.10b) toghether with (2.11b) we can easily verify that

$$\langle q | P) = \left(\frac{1}{2\pi \cos t} \right)^{1.2} \exp - i \left(\frac{qP}{\cos t} + \frac{q^2 + p^2}{2} \tan t + \frac{t \, P^2}{2} \right)$$

transforms any solution of $i\partial\psi/\partial t = -\partial^2\psi/2\partial^2 x^2$ into a solution of $i\partial\psi/\partial t = -\partial^2\psi/2\partial^2 x + x^2/2$. Again, by means of a time-dependent transformation we map a problem in which there are no bound states into a problem in which all states are bound.

4. Nonunitary Representations

In [3], and later in [4] Feinsilver developed connection between (generalized) stochastic processes and Hamiltonian mechanics by an appropriate use of symbolic calculus (or operational analysis). At the basis of it is the fact that one can associate to an integrable system with Hamiltonian $H(p)$ a semigroup with generator $G = H(-\nabla_q)$. As examples of the basic homogeneous (in space and time) processes on $\mathbb{R}$ we have the processes associated with the list (2.12), namely the processes with generators

$$\begin{aligned} G_1 &= a \cdot \nabla, \\ G_2 &= \tfrac{1}{2}\Delta, \qquad\qquad (4.1) \\ G_2 &= \int (e^{\xi \cdot \nabla} - 1) \, \mu(d\xi), \end{aligned}$$

which are, respectively, drift with constant speed a the brownian motion process and a pure jump process with jump distribution given by $\mu(d\xi)$. For arbitrary $H(p)$ we have a semigroup $P_t = \exp Gt$ which may not be positive (hence the name: generalized stochastic process). In this connection see [7] where the class of stochastic processes associated with Hamiltonians of the type $H(p) = p^k$ $k > 2$ is discussed.

The following class of transforms were already introduced in [10, 11], but we recall things for completeness sake.

DEFINITION 1.2. Let f be defined by (2.3). Put

$$(T_F(t)f)(x) = \int \exp - F(q, ik, t)\hat{f}(k)\, dk/(2\pi)^n,$$

where $\hat{f}(k) = \int \exp ik \cdot x f(x)\, dx$ is the Fourier transform of f.

Certainly the class of f' for which T_F is defined must be made more precise. But we assume there are enough such functions.

LEMMA 4.3. *The correspondence $F \to T_F$ is a group homomorphism.*

Proof. It is very simple.

A consequence of this is that the commutative diagram

$$\begin{tikzcd} \mathbb{R}^{2n} \arrow[r, "\bar{S}_1"] \arrow[d, "\mathrm{id}"'] & \mathbb{R}^{2n} \arrow[d] \\ \mathbb{R}^{2n} \arrow[r, "\bar{S}_2"] & \mathbb{R}^{2n} \end{tikzcd} \tag{4.4}$$

has a corresponding diagram

$$\begin{tikzcd} f(\mathbb{R}^n) \arrow[r, "P_1(t)"] \arrow[d, "\mathrm{id}"'] & f(\mathbb{R}^n) \arrow[d, "T_F(t)"] \\ f(\mathbb{R}^n) \arrow[r, "P_2(t)"] & f(\mathbb{R}^n) \end{tikzcd} \tag{4.5}$$

where $\bar{S}_i(q, P, t) = q \cdot H + tH_i(P)$ and $P_i(t) = \exp tG_i$. This is the content of

LEMMA 4.6. *Let $H_i(p)$ generate the semigroups $P_i(t) = \exp tH(-\nabla_{q_i})$, then if $U_i(q, t)$ denote the solutions to*

$$\frac{\partial U_i}{\partial t} = H_i(-\nabla)\, U_i, \qquad U_i(x, 0) = f(x)$$

and if $F = q \cdot P - tH_1(P) + tH_2(P)$, *then*

$$U_1(q, t) = (T_F(t)\, U(\cdot, t))(q).$$

Actually, this lemma is a consequence of the group property of the correspondence $F \to T_F$ and the diagrams above since $(P_i(t) f)(q) = (T_{\bar{S}_i}(t) f)(q)$ as can be easily checked.

5. Moment Systems

Apart of using operational calculus to compute $P_t f(q)$ as $f(C^+(t))\, 1(x)$, where $C^+(t)$ is the operator obtained from $q(t) = q + tv(p)$ upon replacement of p by $-\nabla_q$, in [3, 4]. Feinsilver constructed several classes of special functions associated to each Hamiltonian function $H(p)$. An interesting class of these are the moment systems constructed as follows.

Definition 5.1. For the multi-index $\mathbf{m} = (m_1, ..., m_n)$

$$h_{\mathbf{m}}(q, t) = (C^+(t))^{\mathbf{m}}(q),$$

where as usual, $a^m = a_1^{m_1} \cdots a_n^{m_n}$ for any object a with n components.

We follow the conventional rules: $\binom{\mathbf{m}}{\mathbf{k}} = \binom{m_1}{k_1} \cdots \binom{m_n}{k_n}$, $\mathbf{k} \leqslant m$ if $k_i \leqslant m_i$ for $i = 1, ..., n$; e_i denotes the unit vector with componenets δ_{ij} and $\mathbf{k}! = k_1! \cdots k_n!$

It is shown in [3.4], that:

Proposition 5.2. *Denote by C the operator ∇_q. Since $G = H(-\nabla_q)$ then $C(t) = \exp tG\, C \exp -tG = C$ and we have.*

(a) $[C_i, C_j^+] = \delta_{ij}$,

(b) $C_i^+ h_{\mathbf{m}} = h_{(\mathbf{m} + e_i)}$,

(c) $C_i h_{\mathbf{m}} = m_i h_{(m - e_i)}$,

(d) $C_i^+ C_i, h_{\mathbf{m}} = h_{\mathbf{m}}$,

(e) $\partial h_{\mathbf{m}} / \partial t = G h_{\mathbf{m}}$.

The generating function $g(q, t; a)$ for the $h_{\mathbf{m}}$ is defined by, a being a vector in $\mathbb{R}^n$,

$$g(q, t; a) = \sum_{\mathbf{m}} \frac{a^{\mathbf{m}}}{\mathbf{m}!} h_{\mathbf{m}}. \tag{5.3}$$

From basic operational calculus if follows that

$$g(q, t; a) = (e^{a \cdot C^{+}(t)} 1)(q) = (e^{tG} e^{a \cdot q} e^{-tG} 1)(q) = e^{a \cdot q + tH(-a)}$$

and replacing a by $-ik$ we obtain

$$g(q, t; ik) = \exp -S(q, ik, t),$$

where $S(x, P, t) = x \cdot P - tH(P)$.

In [5] the following problem was explored. Under the action of the group of smooth invertible mappings $U: \mathbb{R}^n \to \mathbb{R}^n$, each Hamilonian has an orbit $\{H(p) = H(U(p)): U \text{ invertible}\}$.

All the elements of the orbit are canonically equivalent and their study is related to the problem of orthonormalization of a moment sequence.

There the canonical transformations involved were static, i.e., time independent, but now we know that by means of a time-independent transformation any two hamiltonian systems and their associated semigroups can be transformed onto each other. And we have

PROPOSITION 5.5. *For the Hamiltonians $H(p)$ and $\tilde{H}(P)$ the generating functions $g(q, t, ik)$ and $\tilde{g}(Q, t; ik)$ defined by* (5.1) *satisfy $\partial g/\partial t = Gg$ and $\partial \tilde{g}/\partial t = \tilde{G}\tilde{g}$, are both equal to* 1 *at $t = 0$ and we also have*

$$g(q, t; ik) = (T_F(t)\, \tilde{g}(\cdot, t; ik))(q)$$

with $F = q \cdot P - tH(P) + t\tilde{H}(P)$.

Proof. The first part is easy and the second follows from Lemma 4.6.

Either by differentiating with respect to k and putting $k = 0$ or proceeding as in (5.5) we can prove that

$$h_{\mathbf{m}}(q, t) = (T_F(t)\, \tilde{h}_{\mathbf{m}}(\cdot, t))(q). \tag{5.6}$$

In particular, the moment sequence associated to a constant Hamiltonian $\tilde{H} = 0$ is $\{Q^{\mathbf{m}}\}$, and it can be canonically transformed any other moment sequence.

These results can be combined with those obtained in [5] to map any two cross-sequences onto each other. Cross-sequences are polynomial sequences $\{J_m(q, t)\}$ with generating functions (see [12] for more) $g(q, a, t) = \exp q \cdot U(a) + tf(a) = \exp q \cdot U(a) + tH(a) = \sum_{\mathbf{m}} (a^{\mathbf{n}}/\mathbf{m})\, J_{\mathbf{m}}(q, t)$. Here $U: \mathbb{R}^n \to \mathbb{R}^n$ is invertible, $V = U^{-1}$ and $H(a) = f(V(a))$.

Putting $a=-ik$ we observe that $\partial g/\partial t=H(-\nabla_q)g$ and if $F(q,P)=q\cdot V(P)$ then

$$(T_F g(\cdot,-ik,t))(q)=\int e^{-Q\cdot V(ik')}\frac{dk'}{(2\pi)^n}\int e^{ik'\cdot Q'}g(Q',-ik;t)\,dQ'$$
$$=\exp\quad -iQ\cdot k+t\tilde{H}(ik).$$

This preamble is the basic ingredient in the proof of

PROPOSITION 5.7. *Let $g_i(q,a;t)=\exp q\cdot U_i(a)+tf_i(a)=\exp\ q\cdot U(a)+tH_i(a)$, $i=1,2$; $H_i(a)=f_i(V_i(a))$ and $V_i=U_i^{-1}$.*

Then by succesively applying the integral transforms associated to the generating functions $F_1=q\cdot V_2(P)$, $F=q\cdot P-tH_1(p)+tH_2(p)$ and $F_2=q\cdot U_1(p)$ the generating function $g_2(q,t;a)$ can be transformed into $g_1(q,a;)t)$, and the same is true for their associated cross-sequences.

FINAL COMMENTS

The equivalence of systems with quadratic hamiltonians has been explored in [1], and in [9] where the time-dependent approach is used but in a different way.

One problem to be solved is the following: how to transfer the canonical equivalence of semigroups to the equivalence of (or transformation law between) the measure and the sample or path spaces associated to each process.

Since we have a "mechanical" approach to polynomials having combinatorial significance, there should be a combinatorial interpretation of the transformation between these polynomials).

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